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ON THE MEASURE OF INTERSECTION OF CYLINDERS

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ABSTRACT

In this paper we study an extremal problem on the minimum of the measure of intersection of cylinders satisfying certain geometric conditions. The main result generalizes previous results of Khatri, Sidak and Gluskin to the wider class of all rotation invariant measures.

Let rD_k denote the k-dimensional disk of radius r in the Euclidean space $\mathbb{R}^n$, i.e. D_k is a centered unit Euclidean ball in some k-dimensional linear subspace. By a **rounded cylinder** we will call a set of the form $rD_k \times \mathbb{R}^{n-k}$, where $\mathbb{R}^{n-k}$ is orthogonal to D_k. Let $O(n)$ denote the orthogonal group in $\mathbb{R}^n$. We call two linear subspaces $M, N \subset \mathbb{R}^n$ pairwise orthogonal in the following sense: if $\dim M + \dim N \leq n$ then M and N are pairwise orthogonal in the usual sense; if $\dim M + \dim N \geq n$ then M and N span all the space $\mathbb{R}^n$ and $(M \ominus (M \cap N))$ is orthogonal to $(N \ominus (M \cap N))$ in the usual sense.

The main results of this paper are Theorems 1 and 2 below.

THEOREM 1: *Let μ be a nonnegative rotation invariant measure on $\mathbb{R}^n$. Let T_1 be a rounded cylinder*

$$T_1 = rD \times E,$$

where $E \subset \mathbb{R}^n$ is a linear subspace and $D \subset E^\perp$ is the unit Euclidean ball in $E^\perp$. Let T_2 be a (not rounded) cylinder of the form

$$T_2 = \Omega \times F,$$

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where $F \subset \mathbb{R}^n$ is a linear subspace and $\Omega \subset F^\perp$ is a centrally symmetric convex body.

Then

$$\min_{U \in O(n)} \mu(U(T_1) \cap T_2)$$

is achieved when the subspaces $\operatorname{span}(UD)$ *and* $\operatorname{span}\Omega$ *are pairwise orthogonal in the above sense.*

THEOREM 2: *Let $T_1, \dots, T_s$ be rounded cylinders,*

$$T_i = r_i D_{k_i} \times \mathbb{R}^{n-k_i},$$

such that $\sum_{i=1}^{s} k_i \leq n$. Let μ be any nonnegative measure on $\mathbb{R}^n$. Then

$$\min_{U_i \in O(n)} \mu\Big(\bigcap_1^s U_i(T_i)\Big)$$

is achieved when all $U_i(T_i)$ are pairwise orthogonal, more precisely $\operatorname{span} U_i(D_{k_i})$ *is orthogonal to* $\operatorname{span} U_j(D_{k_j})$, $i \neq j$.

Remarks: (1) Theorem 2 generalizes the previous results of Khatri [7], Sidak [9] and Gluskin [3], [4]. It follows from [7], [9], [3] that Theorem 2 is true when all cylinders are strips (i.e. $T_i = [-r_i, r_i] \times \mathbb{R}^{n-1}$) and the measure μ is, in addition, absolutely continuous with respect to the Lebesgue measure with *radial decreasing* density. Theorem 2 for arbitrary cylinders follows directly from the argument of [4] in the case when μ is the standard *Gaussian* measure.

(2) A new interesting special case of Theorem 2 is when the measure μ is the Lebesgue measure concentrated on the Euclidean sphere.

(3) Some geometric properties of the intersection of the projective caps (i.e. complements to the strips) on the sphere were studied by Litvak, Milman and Schechtman [8], where applications to the asymptotic theory of Banach spaces can be found.

(4) For the related results on the volume of intersection of caps on the sphere, see the paper by Gromov [5].

Assuming Theorem 1 let us prove Theorem 2.

Proof of Theorem 2: One can easily see that the intersection of the first $(s-1)$ cylinders $T_1 \cap \dots \cap T_{s-1}$ has the form $\Omega \times N_{k_s}$, where N_{k_s} is a linear subspace of dimension $\dim N_{k_s} = k_s$ and Ω is a centrally symmetric convex set contained in the orthogonal complement $N_{k_s}^\perp$.

By Theorem 1, $\min_{U\in O(n)} \mu((\Omega \times N_{k_s}) \cap UT_s)$ is achieved when UD_{k_s} is contained in N_{k_s}. Then an easy induction in s finishes the proof. ∎

Theorem 2 applied to the Lebesgue measure on the unit sphere implies easily the following result (however, as pointed out to us by B. Sudakov and the referee, it has a direct simpler proof).

Corollary 3: *Let $E_1, \dots, E_s$ be linear subspaces of the real (complex) n-dimensional Euclidean (Hermitian) space such that $\sum_1^s \dim(E_i) = n$. Let $w_1, \dots, w_s$ be nonnegative numbers with $\sum_1^s w_i^2 = 1$. Then there exists a vector y from the unit sphere, $\|y\| = 1$, such that*

$$\| \operatorname{Pr}_{E_i} y\| \leq w_i \quad \text{for all } i = 1, \dots, s,$$

where Pr_E denotes the orthogonal projection onto the subspace E.

In particular, if $x_1, \dots, x_n$ are vectors with $\|x_i\| = 1$ and $w_1, \dots, w_n$ are nonnegative numbers with $\sum_1^n w_i^2 = 1$, then there exists y, $\|y\| = 1$, such that

$$|\langle y, x_i\rangle| \leq w_i \quad \text{for all } i = 1, \dots, n,$$

where $\langle \cdot, \cdot \rangle$ is the inner product.

Before proving Theorem 1 we will state the following lemma (which is essentially well-known).

Lemma: *Let $k \geq 1$. Let E_0, E be k-dimensional subspaces of the Euclidean space $\mathbb{R}^{2k}$. Write $\mathbb{R}^{2k} = E_0 \oplus E_0^{\perp}$.*

Then there exists an operator

$$V\colon E_0 \longrightarrow E_0^{\perp}$$

such that $\|V\| \leq \pi/2$ and

$$E = \exp\begin{pmatrix} 0 & -V^* \\ V & 0 \end{pmatrix} E_0.$$

Proof (Compare [6], Ch. I, §1, Theorem 1): Consider the restriction to E of the orthogonal projection onto E_0:

$$P\colon E \longrightarrow E_0.$$

By the general fact of operator theory, there exists an orthonormal basis $x_1, \dots, x_k$ in E_0 and orthonormal basis $y_1, \dots, y_k$ in E such that

$$Py_i = s_i x_i \quad \text{for every } i,$$

where s_i are the singular numbers of P, hence $0 \leq s_i \leq 1$. Let us choose a unit vector x_i' in the span of x_i, y_i to be orthogonal to x_i. Then easily $x_1, x_1', \dots, x_k, x_k'$ form an orthonormal basis in $\mathbb{R}^{2k}$.

Put $\cos\beta_i := s_i$, $0 \leq \beta_i \leq \pi/2$.

Let us define operator $V\colon E_0 \longrightarrow E_0^\perp$ as

$$Vx_i = \beta_i x_i'.$$

Then easily V satisfies the lemma. ∎

Remark: In fact, the curves of the form

$$\exp t \cdot \begin{pmatrix} 0 & -V^* \\ V & 0 \end{pmatrix} E_0$$

are geodesics in the Grassmannian $\mathrm{Gr}_{2k,k}$ and every geodesic through E_0 has such a form (see e.g. [1], Corollary 3.33(2)). But we will not use this property.

Proof of Theorem 1: Let us prove the theorem when the measure μ is the Lebesgue measure concentrated on the unit sphere S^{n-1}, since the general case follows immediately by integration in polar coordinates.

Passing if necessary to smaller n and using Fubini's theorem, we may assume that $\dim D + \dim \Omega \geq n$. We may and will also assume that D and Ω are in generic position, namely if

$$L := \operatorname{span} D \cap \operatorname{span} \Omega,$$

then $\dim L = \dim(\operatorname{span} D) + \dim(\operatorname{span} \Omega) - n$.

Consider another rounded cylinder

$$T = r\tilde{D} \times G,$$

where $\tilde{D}$ is the unit ball in the linear space $G^\perp$ (so that $\operatorname{span} \tilde{D} = G^\perp$),

$$\dim \tilde{D} = \dim D,$$

$$\operatorname{span} \tilde{D} \supset L,$$

and $\operatorname{span} \tilde{D}$ is orthogonal to $\operatorname{span} \Omega$ in the sense mentioned at the beginning of the paper. We want to prove that

$$\mu(T \cap T_2) \leq \mu(T_1 \cap T_2).$$

Next we observe that the case $\dim D > \dim \Omega$ can be easily reduced to the case $\dim D \leq \dim \Omega$ by lowering the dimension of the total space. Indeed, set

$$H = (\operatorname{span} \tilde{D}) \cap (\operatorname{span} D),$$

$$J = (\operatorname{span} \tilde{D}) \ominus H.$$

Easily $\dim J + \dim L < \dim(\operatorname{span} \tilde{D})$. Then

$$V := (\operatorname{span} \tilde{D}) \ominus (L \oplus J)$$

is non-trivial, $V \perp \operatorname{span} \Omega$, and $V \subset H$. Then

$$\mathbb{R}^n = (\operatorname{span} \Omega \oplus J) \oplus V.$$

Using Fubini's theorem for sections of S^{n-1} parallel to $(\operatorname{span} \Omega \oplus J)$ we see that the case $\dim D > \dim \Omega$ is reduced to the case $\dim D \leq \dim \Omega$. So let us assume that

$$\dim D \leq \dim \Omega.$$

Note that since D and Ω are assumed to be in generic position, then

$$L = (\operatorname{span} D) \cap (\operatorname{span} \Omega) = (\operatorname{span} \tilde{D}) \cap (\operatorname{span} \Omega).$$

Let $\Gamma = (\operatorname{span} \tilde{D}) \ominus L$. By the trivial linear algebra argument one can decompose $\operatorname{span} \Omega$ into an orthogonal direct sum

$$\operatorname{span} \Omega = L \oplus M \oplus \Delta,$$

where $\dim \Delta = \dim \Gamma$, $L \oplus \Delta \oplus \Gamma \supset \operatorname{span} D$, and M is just an orthogonal complement of $L \oplus \Delta$ inside $\operatorname{span} \Omega$. Thus

$$\mathbb{R}^n = L \oplus M \oplus \Delta \oplus \Gamma, \tag{1}$$

$$\Omega \subset L \oplus M \oplus \Delta \oplus \{0\} \quad \text{and} \quad T \subset L \oplus \{0\} \oplus \{0\} \oplus \Gamma. \tag{2}$$

Using Lemma 4 one can find an operator

$$V \colon \Gamma \longrightarrow \Delta$$

such that

$$\|V\| \leq \frac{\pi}{2}, \tag{3}$$

$$\exp \left(\begin{array}{c|cc} 0 & \multicolumn{2}{c}{0} \\ \hline 0 & 0 & V \\ & -V^* & 0 \end{array} \right) (\operatorname{span} \tilde{D}) = \operatorname{span} D. \tag{4}$$

Consider a one-parametric group of *orthogonal* operators A_t: $\mathbb{R}^n \longrightarrow \mathbb{R}^n$, $0 \le t \le 1$, defined by

$$A_t = \exp t \cdot \left(\begin{array}{c|cc} 0 & \multicolumn{2}{c}{0} \\ \hline 0 & 0 & V \\ & -V^* & 0 \end{array}\right). \tag{5}$$

In particular, A_t acts as the identity operator on $L \oplus M$ for all t. Clearly $A_0 = \mathrm{Id}_n$ and $A_1 T = T_1$. To prove Theorem 1 it suffices to show that

$$\mathrm{mes}(S^{n-1} \cap A_t T \cap (\Omega \times F))$$

increases in $t \in [0,1]$, where mes denotes the Lebesgue measure concentrated on the unit sphere S^{n-1} (recall that $T_2 = \Omega \times F$).

First recall that a characteristic (indicator) function of every compact convex centrally symmetric set Ω can be approximated a.e. and in the L_1-norm by a smooth log-concave even function (to see this consider the convolution

$$f_\varepsilon = \mathbb{1}_\Omega * \left(\frac{1}{(\sqrt{2\pi\varepsilon})^n} e^{-|x/\varepsilon|^2/2} \right),$$

where $\mathbb{1}_\Omega$ is the characteristic function of Ω, $\varepsilon \to +0$. Then f_ε is a log-concave function by the Prekopa–Leindler inequality as a convolution of two log-concave functions, and $f_\varepsilon \to \mathbb{1}_\Omega$ a.e. and in L_1-norm). In our case $\Omega \subset L \oplus M \oplus \Delta \oplus \{0\}$, hence taking the above convolution with respect to this subspace we may assume that for $p = (X_1, X_2, Y, Z)$, $f_\varepsilon(p) = f_\varepsilon(X_1, X_2, Y)$ (where the coordinates correspond to the decomposition (1)). It is easy to see that if we restrict all the functions onto the unit sphere $S^{n-1} \subset \mathbb{R}^n$, then $f_\varepsilon|_{S^{n-1}} \longrightarrow \mathbb{1}_\Omega|_{S^{n-1}}$ a.e. and in $L_1(S^{n-1})$. Hence in order to prove the theorem it is sufficient to show that if $f(p) = f(X_1, X_2, Y)$ is a smooth log-concave even function, then (setting $T' := T \cap S^{n-1}$)

$$\varphi(t) := \int_{A_t T'} f(p) dp, \quad 0 \le t \le 1$$

is an increasing function of t.

One can easily check the following formula (since A_t are measure preserving):

$$\frac{d}{dt}\varphi(t) = \frac{d}{dt} \int_{T'} f(A_t p) dp = \int_{\partial T'} f(A_t p) \langle n(p), v(p,t) \rangle dp, \tag{6}$$

where ∂T ($\subset S^{n-1}$) is the boundary of T', $n(p)$ is the unique *inner* normal to $\partial T'$ at $p \in \partial T'$ (in particular, $n(p)$ is a tangent vector to S^{n-1} at p), and

$$v(p,t) = \frac{d}{dt} A_t p$$

is the vector of velocity at point p at time t.

Clearly,

$$(7)\ \partial T' = \{(X_1, X_2, Y, Z) \in S^{n-1} \big| |X_1|^2 + |Z|^2 = r^2\} = \sqrt{1-r^2} S_{X_2,Y} \times r S_{X_1,Z}$$

where $|\cdot|$ denotes the Euclidean norm, $S_{X_1,Z}$ is the unit sphere in $L \oplus \Gamma$, and $S_{X_2,Y}$ is the unit sphere in $M \oplus \Delta$. We may assume $r < 1$. Then a direct elementary computation shows that, at $p = (X_1, X_2, Y, Z) \in \partial T'$,

$$(8)\qquad n(p) = \left(-\frac{\sqrt{1-r^2}}{r} X_1, \frac{r}{\sqrt{1-r^2}} X_2, \frac{r}{\sqrt{1-r^2}} Y, -\frac{\sqrt{1-r^2}}{r} Z \right).$$

Since

$$\frac{d}{dt} A_t = \left(\begin{array}{c|cc} 0 & \multicolumn{2}{c}{0} \\ \hline 0 & 0 & V \\ & -V^* & 0 \end{array} \right) \cdot A_t,$$

then obviously

$$(9)\qquad v(p,t) = (0, 0, VZ, -V^*Y).$$

Substituting (7), (8), (9) into (6) we obtain

$$\begin{aligned} \frac{d}{dt}\varphi(t) &= \int_{\partial T'} f(A_t p) \left[\frac{r}{\sqrt{1-r^2}} \langle Y, VZ \rangle + \frac{\sqrt{1-r^2}}{r} \langle Z, V^*Y \rangle \right] \\ (10)\qquad &= \frac{1}{r\sqrt{1-r^2}} \int_{rS_{X_1,Z}} dX_1 dZ \int_{\sqrt{1-r^2} S_{X_2,Y}} dX_2 dY f(A_t p) \langle Y, VZ \rangle. \end{aligned}$$

Let us compute A_t. First

$$\begin{aligned} \exp \begin{pmatrix} 0 & V \\ -V^* & 0 \end{pmatrix} &= \sum_{k=0}^{\infty} \frac{1}{k!} \begin{pmatrix} 0 & V \\ -V^* & 0 \end{pmatrix}^k \\ &= \sum_{j=0}^{\infty} \left[\frac{1}{(2j)!} \begin{pmatrix} 0 & V \\ -V^* & 0 \end{pmatrix}^{2j} + \frac{1}{(2j+1)!} \begin{pmatrix} 0 & V \\ -V^* & 0 \end{pmatrix}^{2j+1} \right]. \end{aligned}$$

Since

$$\begin{pmatrix} 0 & V \\ -V^* & 0 \end{pmatrix}^2 = -\begin{pmatrix} VV^* & 0 \\ 0 & V^*V \end{pmatrix},$$

the last expression is equal to

$$\sum_{j=0}^{\infty}\left[\frac{1}{(2j)!}(-1)^j\begin{pmatrix}(VV^*)^j & 0\\ 0 & (V^*V)^j\end{pmatrix}\right.$$
$$\left.+\frac{1}{(2j+1)!}(-1)^j\begin{pmatrix}(VV^*)^j & 0\\ 0 & (V^*V)^j\end{pmatrix}\begin{pmatrix}0 & V\\ -V^* & 0\end{pmatrix}\right]$$
$$(11)\qquad =\sum_{j=0}^{\infty}\left[\frac{1}{(2j)!}(-1)^j\begin{pmatrix}(\sqrt{VV^*})^{2j} & 0\\ 0 & (\sqrt{V^*V})^{2j}\end{pmatrix}\right.$$
$$\left.+\frac{1}{(2j+1)!}(-1)^j\begin{pmatrix}0 & (VV^*)^jV\\ -(V^*V)^jV^* & 0\end{pmatrix}\right].$$

Using (11) and the identities

$$\sum_{j=0}^{\infty}\frac{1}{(2j)!}(-1)^jx^{2j}\equiv\cos x,\quad \sum_{j=0}^{\infty}\frac{1}{(2i+1)!}(-1)^jx^{2j}\equiv\frac{\sin x}{x},$$

and denoting for brevity $\lambda(x):=(\sin x)/x$, we obtain

$$\exp\begin{pmatrix}0 & V\\ -V^* & 0\end{pmatrix}=\begin{pmatrix}\cos\sqrt{VV^*} & \lambda(\sqrt{VV^*})V\\ -\lambda(\sqrt{V^*V})V^* & \cos\sqrt{V^*V}\end{pmatrix},$$

where $\lambda\ (\sqrt{V^*V})$ denotes the function $(\sin x)/x$ of the operator $\sqrt{V^*V}$. Hence if $p=(X_1,X_2,Y,Z)$, then

$$A_t(p)=(X_1,X_2,\cos(t\sqrt{VV^*})Y+\lambda(t\sqrt{VV^*})tVZ,*),$$

where * denotes some expression, which is irrelevant for us. It follows from this and (10) that

$$r\sqrt{1-r^2}\frac{d}{dt}\varphi(t)=\int_{rS_{X_1,Z}}dX_1dZ\int_{\sqrt{1-r^2}S_{X_2,Y}}f\Big(X_1,X_2,\cos(t\sqrt{VV^*})Y$$
$$(12)\qquad +\lambda(t\sqrt{VV^*})tVZ\Big)\langle Y,VZ\rangle dX_2dY.$$

We want to show that the expression (12) is *nonnegative*. Let us fix $0<t<1$. Since $\|V\|\le\pi/2$, the operator $\cos(t\sqrt{VV^*})$ is selfadjoint and invertible. Consider a new smooth function on $L\oplus M\oplus\Delta$:

$$g(X_1,X_2,Y):=f(X_1,X_2,\cos(t\sqrt{VV^*})Y),$$

which is obviously *log-concave* and *even*. Consider a selfadjoint operator A: $\Delta \to \Delta$,

$$A := t \cdot \lambda(t\sqrt{VV^*}) \Big/ \cos(t\sqrt{VV^*}). \tag{13}$$

Since $\|V\| \le \pi/2$, $0 < t < 1$ and

$$\frac{\sin x}{x \cos x} > 0 \quad \text{if } |x| < \frac{\pi}{2},$$

then $A > 0$. Then the right hand side of (12) can be rewritten

$$\int_{rS_{X_1,Z}} dX_1 dZ \int_{\sqrt{1-r^2}S_{X_2,Y}} g(X_1, X_2, Y + AVZ)\langle Y, VZ\rangle dX_2 dY. \tag{14}$$

Using the same trick as previously (but now for functions not on the sphere, but on the Euclidean space) and denoting $B_{X_2,Y}$ the unit Euclidean ball in $M \oplus \Delta$, we have for the inner integral in (14)

$$\begin{aligned}
&\frac{1}{\sqrt{1-r^2}} \cdot \int_{\sqrt{1-r^2}S_{X_2,Y}} g(X_1, X_2, Y + AVZ)\langle Y, VZ\rangle dX_2 dY \\
&= -\frac{d}{d\tau}\Big|_{\tau=0} \int_{\sqrt{1-r^2}B_{X_2,Y}} g(X_1, X_2, Y + AVZ + \tau VZ) dX_2 dY \qquad (15)\\
&= -\frac{d}{d\tau}\Big|_{\tau=0} \int_{\sqrt{1-r^2}B_{X_2,Y}} g(X_1, X_2, A(A^{-1}Y + VZ + \tau A^{-1}VZ)) dX_2 dY \\
&= -\frac{d}{d\tau}\Big|_{\tau=0} \int_{\sqrt{1-r^2}B_{X_2,Y}} h(X_1, X_2, A^{-1}Y + VZ + \tau A^{-1}VZ) dX_2 dY,
\end{aligned}$$

where $h(X_1, X_2, Y) = g(X_1, X_2, AY)$. By (13), A^{-1} can be represented in the form

$$A^{-1} = \sum_{k \ge 0} a_k (VV^*)^k,$$

where a_k are the coefficients in the power series of $\cos(tx) \big/ t \cdot \lambda(tx)$. Thus

$$\begin{aligned}
A^{-1}V &= \Big(\sum_k a_k (VV^*)^k\Big) V \\
&= V \sum_k a_k (V^*V)^k \\
&= V \cdot \cos(t\sqrt{V^*V}) \Big/ t \cdot \lambda(t\sqrt{V^*V}) \\
&= V \cdot B, \qquad (16)
\end{aligned}$$

where B is a selfadjoint *positive* operator. Define on $L \oplus \Gamma$ a function

$$\theta(X_1, Z) = \int\limits_{\sqrt{1-r^2}B_{X_2,Y}} h(X_1, X_2, A^{-1}Y + VZ)dX_2dY. \tag{17}$$

Recall that our theorem would follow from non-negativity of (14). So in this notation we have to prove that

$$\frac{d}{d\tau}\Big|_{\tau=0} \int\limits_{rS_{X_1,Z}} \theta(X_1, Z + \tau BZ)dX_1dZ \leq 0. \tag{18}$$

Since h is log-concave, θ is also log-concave, as follows from the Prekopa–Leindler inequality. Moreover, since h is smooth and even, hence θ is also smooth and even. Then (18) follows from the next lemma.

LEMMA 5: *Let θ be a smooth even log-concave function on $\mathbb{R}^k$. Let $B\colon \mathbb{R}^k \to \mathbb{R}^k$ be a selfadjoint operator, $B \geq 0$. Then*

$$\frac{d}{d\tau}\Big|_{\tau=0} \int\limits_{S^{k-1}} \theta(W + \tau BW)dW \leq 0.$$

Proof: Observe that

$$\frac{d}{d\tau}\Big|_{\tau=0} \int\limits_{S^{k-1}} \theta(W + \tau BW)dW = \frac{d}{d\tau}\Big|_{\tau=0} \int\limits_{S^{k-1}} \theta(e^{\tau B}W)dW.$$

It remains to check that the function of τ,

$$\int\limits_{S^{k-1}} \theta(e^{\tau B}W)dW,$$

decreases. Every even log-concave function can be approximated by a linear combination with positive coefficients of characteristic functions of convex centrally symmetric sets. Let K be such a set $\mathbb{1}_K$ be its characteristic function. Then

$$\int\limits_{S^{k-1}} \mathbb{1}_K(e^{\tau B}W)dW = \int\limits_{S^{k-1}} \mathbb{1}_{e^{-\tau B}K}(W)dW = \mathrm{mes}(S^{k-1} \cap e^{-\tau B}K).$$

Since $B \geq 0$, then $e^{-\tau B}$ is a contraction, i.e. $\|e^{-\tau B}\| \leq 1$.

Now the lemma follows from the result of [2], which states that if K is a centrally symmetric convex body, $C\colon \mathbb{R}^k \longrightarrow \mathbb{R}^k$ a contraction, then

$$\mathrm{mes}(S^{k-1} \cap CK) \leq \mathrm{mes}(S^{k-1} \cap K). \qquad \blacksquare$$

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